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# On soliton creation in the nonlinear Schrödinger models: discrete and continuous versions

V V Konotop† and V E Vekslerchik

Institute for Radiophysics and Electronics, Academy of Sciences of Ukraine, Proscura Street 12, Kharkov 310085, Ukraine

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**Abstract.** The creation of new solitons of both the nonlinear Schrödinger equation and its discrete analogy caused by perturbations of one-soliton initial conditions is considered within the framework of the inverse scattering technique.

## 1. Introduction

As is known, the creation of dark solitons is a thresholdless process, which for the first time, to all appearance, was observed experimentally in optics by Krokel *et al* [1] and then theoretically investigated by Gredescul and Kivshar [2]. In our previous paper [3] where dynamics of perturbed dark solitons within the framework of the nonlinear Schrödinger equation (NSE) was discussed, we investigated the creation of small additional solitons due to perturbations of initial conditions corresponding to the pure one-soliton pulses. In the present paper we wish first to generalize some results of [3], namely, the dispersion relation leading to the discovery of a number of solitons was obtained there for a particular case of the so-called black soliton (i.e. having zero velocity). Here we derive and solve it for any dark soliton solution. Another aim is to investigate this problem in the framework of the discrete version of the NSE (DNSE) which was introduced by Ablowitz and Ladik [4]. The DNSE, having a rather wide area of application, is also important in view of the fact that a great deal of theoretical investigations of most NSE applications are based on numerical simulations, and hence on a discrete version.

## 2. Statement of the problem

We study the NSE

$$iq_t + q_{xx} + 2(\rho^2 - |q|^2)q = 0 \tag{1}$$

and the DNSE

$$i(q_n)_t + (1 - |q_n|^2)(q_{n-1} + q_{n+1} - 2q_n) + 2(\rho^2 - |q_n|^2)q_n = 0. \tag{2}$$

† Present address: Departamento de Fisica Teorica I, Facultad de Ciencias Fisicas, Universidad Complutense, Ciudad Universitaria E-28040, Madrid, Spain.

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As is clear, equation (1) is a continuous limit ($\delta \to 0$; $\delta$ being the discretization parameter) of equation (2) after the substitution

$$q_n \to q\delta \qquad n\delta \to x \qquad t\delta^2 \to t \qquad \rho \to \rho\delta$$

or, in other words, equation (2) is a discrete version of equation (1).

Since we are dealing with the dark soliton case, these equations will be treated under the boundary conditions

$$\lim_{x\to-\infty} q = \rho \qquad \lim_{x\to\infty} q = \rho\, e^{i\theta} \tag{3}$$

$$\lim_{n\to-\infty} q_n = \rho \qquad \lim_{n\to\infty} q_n = \rho\, e^{i\theta}. \tag{4}$$

In order to study the problem of soliton creation from a slightly modulated dark soliton, consider the following initial conditions:

$$q(x, t=0) = q^{(s)}(x) + \delta q(x) \tag{5}$$

$$q_n(t=0) = q_n^{(s)} + \delta q_n \tag{6}$$

where $q^{(s)}(x)$ and $q_n^{(s)}$ are pure soliton conditions:

$$q^{(s)}(x) = \rho \frac{1 + e^{i\theta} \exp(\nu x)}{1 + \exp(\nu x)} \tag{7}$$

with $\nu = 2\rho \sin(\theta/2)$, and

$$q_n^{(s)} = \rho \frac{1 + e^{i\theta} h^n}{1 + h^n} \tag{8}$$

with $h$ defined by the relation

$$h^{1/2} - h^{-1/2} = 2(\rho/r)\sin(\theta/2) \qquad h > 1 \tag{9}$$

where $r = (1-\rho^2)^{1/2} > 0$ and only the case of $\rho^2 < 1$ is considered.

The one-soliton solution of the NSE can be found, e.g. in [3, 5], while the dark soliton of the DNSE is given by

$$q_n^{(s)}(t) = \rho \frac{1 + e^{i\theta} h^n \exp(\mu t)}{1 + h^n \exp(\mu t)} \tag{10}$$

with $\mu = 2\rho^2 \sin\theta$ (it is a particular case of the one-soliton solution of the discrete Hirota equation recently reported by Narita [6]).

The differences $\delta q(x)$ and $\delta q_n$, describing perturbations of initial soliton pulses, are assumed to be sufficiently small (the precise meaning of this is discussed below).

What we are interested in is the number of solitons formed from the initial pulses given by (5) and (6). That can be obtained by means of the inverse scattering technique (IST) which is applicable to both the NSE and the DNSE. The IST for the NSE has been described in detail in numerous works (see e.g. the monograph [5]) and was outlined in [3]. As to the DNSE, some points of the IST for the case of non-zero boundary conditions will be represented here.

## 3. Linear scattering problem

The linear spectral problems associated with equations (1) and (2) are as follows:

$$(\partial/\partial x)\Phi(x,\lambda) = U(x,\lambda)\Phi(x,\lambda) \tag{11}$$

$$\Phi(n+1, z) = U_n(z)\Phi(n, z). \tag{12}$$

For the definition of the matrix $U$ from (11) see references [3, 5], while the matrix $U_n(z)$ is given by

$$U_n(z)=\begin{pmatrix} z & \bar{q}_n \\ q_n & z^{-1} \end{pmatrix} \tag{13}$$

with the potential $q_n=q_n(t=0)$, $\lambda$ and $z$ being spectral parameters (a bar denotes complex conjugation). The scattering matrices, $\boldsymbol{T}_{c,d}$ (the subscripts stand for continuous and discrete models respectively) are defined by

$$\Phi_-(x,\lambda)=\Phi_+(x,\lambda)\boldsymbol{T}_c(\lambda) \tag{14}$$

$$\Phi_-(n,z)=\Phi_+(n,z)\boldsymbol{T}_d(z). \tag{15}$$

The matrix Jost functions $\Phi_\pm$ for the continuous case were defined in [3, 5]. Their discrete analogies are the solutions of (12), satisfying the limiting conditions

$$\lim_{n\to\pm\infty}\Phi_\pm(n,z)r^{-n}\operatorname{diag}(\zeta^{-n},\zeta^n)=C_\pm(z) \tag{16}$$

where $\zeta$ is defined by

$$r(\zeta+\zeta^{-1})=z+z^{-1} \tag{17}$$

the matrices $C_\pm$ are given by

$$C_-=\begin{pmatrix} 1 & -\eta \\ \eta & 1 \end{pmatrix} \qquad C_+=\begin{pmatrix} \bar{\varepsilon} & -\bar{\varepsilon}\eta \\ \varepsilon\eta & \varepsilon \end{pmatrix} \tag{18}$$

with $\eta=\rho^{-1}(r\zeta-z)$, $\varepsilon=\exp(\theta/2)$.

Characteristics of solitons can be obtained from the discrete spectra of the scattering problems which are defined as the set of zeros of $T_c^{(11)}(\lambda)$ for the case of the NSE (hereafter the designation $a_c(\lambda)=T_c^{(11)}(\lambda)$ is used) and the set of zeros of $T_d^{(22)}(z)$ for the DNSE. For the further purposes it is convenient to introduce the quantity $\xi$ defined by

$$\xi=\frac{\zeta(z)}{z}. \tag{19}$$

In terms of $\xi$ the diagonal part of the scattering matrix is a meromorphic function. The element $T_d^{(22)}$ considered as a function of $\xi$ will be denoted hereafter as $a_d(\xi)$.

Now the relation between parameters of solitons and the eigenvalues of the scattering problems may be expressed as follows. In the continuous case the amplitude $\nu_k$ (which is, at the same time, the inverse width) and the velocity $v_k$ of the $k$th soliton are given by [5]

$$v_k=\lambda_k \qquad \nu_k=(4\rho^2-\lambda_k^2)^{1/2} \tag{20}$$

where $\lambda_k$ is a zero of $a_c(\lambda)$ from the segment $(-2\rho,2\rho)$ of the real axis. In the discrete case, parameters $h_k$ and $\mu_k$ of the $k$th soliton are given by [7]

$$h_k=|\xi_k|^{-2} \qquad \mu_k=2r(h_k-1)\operatorname{Im}\xi_k \tag{21}$$

where $\xi_k$ is a zero of $a_d(\xi)$ from the interior of the unit circle: $|\xi_k|<1$.

The Jost coefficients $a_c(\lambda)$ and $a_d(\xi)$ can be obtained in an explicit form in the pure soliton case (put $\delta q$ and $\delta q_n$ in (5), (6) to be equal to zero):

$$a_c^{(s)}(\lambda)=\varepsilon\frac{\lambda+\sqrt{\lambda^2-4\rho^2}+2\rho\bar{\varepsilon}}{\lambda+\sqrt{\lambda^2-4\rho^2}+2\rho\varepsilon} \tag{22}$$

and

$$a_{\rm d}^{(\rm s)}(\xi) = \varepsilon \frac{\xi - \xi_{\rm s}}{\xi h^{-1} - \xi_{\rm s}} \tag{23}$$

with

$$\xi_{\rm s} = r \frac{\exp({\rm i}\theta) - h^{-1}}{\exp({\rm i}\theta) - 1}. \tag{24}$$

## 4. Perturbation analysis

The case of small deviations $\delta q$ and $\delta q_n$ may be treated using the perturbation technique, based on expansion of the scattering data in functional series in initial perturbations [3]. Omitting the straightforward but rather cumbersome calculations, we present here the resulting formulae:

$$\frac{\delta a_{\rm c}}{a_{\rm c}^{(\rm s)}} = \frac{2{\rm i}}{(\lambda^2 - 4\rho^2)^{1/2}} \left[ \Delta_{\rm r}^{(\rm c)} + \frac{2\rho}{\lambda - \lambda_{\rm s}} \Delta_{\rm i}^{(\rm c)} \right] \tag{25}$$

where $\lambda_{\rm s} = -2\rho \cos(\theta/2)$,

$$\Delta_{\rm r}^{(\rm c)} = {\rm Re} \int_{-\infty}^{\infty} {\rm d}x \, q_{\rm s}(x) \delta \bar{q}(x) \tag{26}$$

$$\Delta_{\rm i}^{(\rm c)} = \frac{1}{2\rho} {\rm Im} \int_{-\infty}^{\infty} {\rm d}x \frac{\rm d}{{\rm d}x} q_{\rm s}(x) \delta \bar{q}(x) \tag{27}$$

for the continuous version (these formulae generalize the expressions (70)-(71) from [3]), and

$$\frac{\delta a_{\rm d}}{a_{\rm d}^{(\rm s)}} = -\frac{1}{r^3} \frac{\xi(\xi - r)}{(\xi - r)^2 + \rho^2} [\Delta_{\rm r}^{(\rm d)} + \Omega(\xi) \Delta_{\rm i}^{(\rm d)}] \tag{28}$$

where

$$\Delta_{\rm r}^{(\rm d)} = \sum_{n=-\infty}^{\infty} \frac{(1 + A_n)^2}{(1 + A_{n-1})(1 + A_{n+1})} {\rm Re}\, q_n^{(\rm s)} \delta \bar{q}_n \tag{29}$$

$$\Delta_{\rm i}^{(\rm d)} = \frac{1}{4\rho^2} \sum_{n=-\infty}^{\infty} {\rm Im}(q_{n+1}^{(\rm s)} - q_{n-1}^{(\rm s)}) \delta \bar{q}_n \tag{30}$$

and

$$\Omega(\xi) = \frac{4{\rm i}\rho}{h + 1} \frac{\xi - r^{-1}}{(\xi - \xi_{\rm s})(\xi \bar{\xi}_{\rm s} - 1)} \tag{31}$$

for the discrete one.

From (25) and (28) it follows that $\delta a_{\rm c}$ and $\delta a_{\rm d}$ possess singular points $\lambda = \pm 2\rho$ and $\xi = r \pm {\rm i}\rho$ respectively, which are the edges of the continuous spectrum (in the discrete case (see (12)) the continuous spectrum is a union of the two arcs of the unit circle ($|z| = 1$, ${\rm Re}\, z < r$), while in terms of $\xi$ it is the unit circle with the two points, $r \pm {\rm i}\rho$, deleted). That the scattering data are singular is a characteristic feature of the case of the non-zero boundary conditions (3), (4), the pure $N$-soliton case (i.e. the case of the reflectionless potentials in (14), (15)) being, apparently, the only case when these singularities are absent [5].

Considering the validity of the approach, it should be pointed out that the exact limitations for the initial perturbation can be obtained from the evident requirement. The next term (i.e. a quadratic functional of $\delta q(x)$ or $\delta q_n$) in the expansion of the Jost coefficient into the functional Taylor series has to be much less than those given by (25)-(31). Corresponding calculations are both quite simple and bulky, and that is why they are not given here. However, as is clear, an approximate requirement for the approach to be valid can be given by $\Delta_{i,r}^{(c)} \ll 1$ or $\Delta_{i,r}^{(d)} \ll 1$ (for continuous and discrete versions correspondingly.

Equating $a^{(s)} + \delta a$ with zero, the dispersion relations can be immediately obtained by

$$(\lambda - \lambda_s)\sqrt{4\rho^2 - \lambda^2} + 2(\lambda - \lambda_s)\Delta_r^{(c)} + 4\rho\Delta_i^{(c)} = 0 \tag{32}$$

for the continuous case (cf equation (72) in [3]), and

$$[(\xi - r)^2 + \rho^2](\xi - \xi_s) - r^{-3}\xi(\xi - r)(\xi - \xi_s)\Delta_r^{(d)} + \frac{4i\rho}{h+1}\frac{\xi - r^{-1}}{\xi\bar{\xi}_s - 1}\Delta_i^{(d)} = 0 \tag{33}$$

for the discrete one.

The analysis of these equations leads to the following result. There are the roots of (32), (33) which are close to the 'unperturbed' ones, $\lambda_s$ and $\xi_s$ (the solitons that correspond to these roots may be called, by analogy with [3], 'prime' solitons). Also, there can be solutions of (32) and (33) lying near the edges of the continuous spectra $\pm 2\rho$ and $r \pm i\rho$. The corresponding solitons (such were termed 'additional' in [3]) are of small amplitudes ($\nu_k \approx 0$, $h_k \approx 1$) and of large velocities. Their number (if any) depends on the relation between $\Delta_{i,r}$.

Seeking solutions $\lambda_\pm$ of equation (32) being close to the points $\pm 2\rho$, they may be joined to the segment $(-2\rho, 2\rho)$ if the requirement

$$\left(1 \pm \cos\frac{\theta}{2}\right)\Delta_r^{(c)} \pm \Delta_i^{(c)} < 0 \tag{34}$$

is satisfied. By analogy, the solutions $\xi_\pm$ of (33) close to the points $r \pm i\rho$ are of physical interest ($|\xi_\pm| < 1$), if

$$rg\left(g \pm \cos\frac{\theta}{2}\right)\Delta_r^{(d)} \pm \Delta_i^{(d)} < 0 \tag{35}$$

where $g = \frac{1}{2}(h^{1/2} + h^{-1/2})$.

## 5. Conclusion

Thus, the main results may be summarized by means of the diagram sketched in figure 1 (both the continuous and discrete cases can be demonstrated simultaneously, with the indices 'c' and 'd' being omitted). The plane $(\Delta_r, \Delta_i)$ can be divided into four sectors in compliance with the number of solitons being created from the initial pulse (the 'two-soliton' sectors 2 and 4 correspond to opposite moving directions of the additional soliton). That all sectors meet at the point $\Delta_i = \Delta_r = 0$ is an illustration of the thresholdless character of dark soliton creation.

Finally, let us review the results in view of the computational application of the DNSE. In the continuous limit $(\delta \to 0)$ $r = 1 + O(\delta^2)$ and $g = 1 + O(\delta^2)$. Hence the discretization can affect the number of solitons calculated only when $\Delta_{i,r}$ are rather

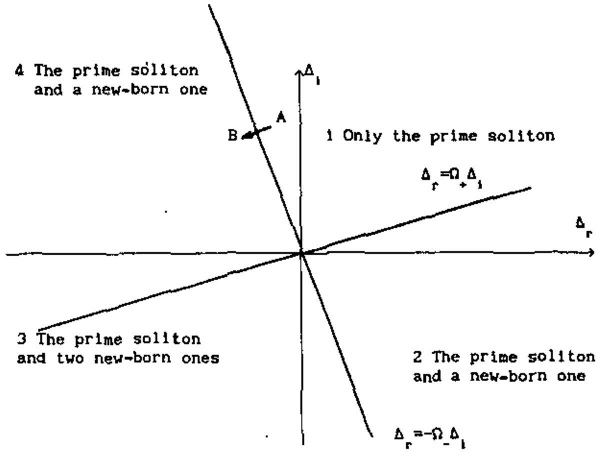


**Figure 1.** The complex plane of $(\Delta_r, \Delta_i)$ is divided into four parts by two lines (for clarity they are depicted for $\theta \in [0, 2\pi]$) crossing zero. The parts correspond to different numbers of solitons. The transition between points $A$ and $B$ is a graphical representation of the system sensibility with respect to small variations of the discreteness parameter. If $A$ is close to one of the dividing lines, even a small change of $\delta$ ($|AB| = O(\delta)$) may result in a change of the number of solitons. Factors $\Omega_\pm$ are defined by $\Omega_\pm = 1 \mp \cos\theta$.

close to the borderlines between the sectors in figure 1, since the difference between $\Delta_{i,r}^{(d)}$ and $\Delta_{i,r}^{(c)}$ is of the order of $\delta$.

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